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R.C MATRICES – 3 X 3 ↔ N X N, GRAPHS, SALIENT FEATURES, AND EIGEN VALUES

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ABSTRACT

R.C matrix is a square matrix in which i^{th} row, for all integer value of i, is orthogonal to i^{th} column. Though the set of all R.C matrices is a sub-class of square matrices but refrains to obey some of the basic tenets of matrix algebra. Member matrices of the set of R.C matrices, except the null matrix, are always non-singular and they disguise many invincible characteristics seemingly uncommon in nature.

KEYWORDS: *R.C Matrix, Eigen Values, Characteristic Polynomial*

INTRODUCTION

Abbreviations

(1) R.C Matrix (Row. Column matrix) **(2) JJn (**The Set of all R.C matrices where 'n' stands for a positive integer greater than) (3) **0** (Null Matrix**)**

ASSUMPTIONS

Any member matrix of the set **JJ3** will follow all fundamental tenets of matrix algebra.

INTRODUCTION

We introduce R.C matrix and try to unveil some of its excellent salient features. These are the features which have become known to us on looking at its basic structure from different angles.

Defining Property and General Form

What -we call, an R.C matrix is an outcome of an open discussion on the various properties of matrices. Just a thought; what can happen if each one of the n rows of a matrix is orthogonal to the corresponding column of the same matrix. In general i^{th} row, for all integer value of i, is orthogonal to an i^{th} column and hence truly justified to name as 'R.C ' matrix.

We give an illustration;

A = $\begin{pmatrix} 1 & 2 & -3 \\ 1 & 2 & -6 \\ 1 & 1 & 3 \end{pmatrix}$ is an R.C matrix.

We write its general form. An R.C matrix A of order nxn from the set **JJn** is as follows.

$$A = \begin{pmatrix} a_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_2 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_3 & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_n \end{pmatrix} \in JJn$$

We have the R.C property; $(a_i)^2 + \sum_{j=1}^{j=n} a_{ij}a_{ji} = 0 \; for \; \boldsymbol{i} \neq \boldsymbol{j}, i = 1 \; to \; n$ (1)

[e.g. $(a_1)^2 + a_{12}\, a_{21} + a_{13}\, a_{31} + a_{14}\, a_{41} + \ldots\ldots\ldots = 0$]

We have R.C matrices of higher order and even can be constructed by extending R.C property from a given R.C matrix of order 3x3.

R.C matrices of order 2x2 are (1) $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, (2) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, (3) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, *and* (4) $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ (2)

These are parallel in constitutional nature to what is known as spin matrices in quantum physics. This will give our readers the best insight giving the debut in the vast field.

PairWise Graphical Presentation

At this stage, we would prefer to draw facts from graphical presentation of any R.C matrix and extend our imaginations to possibly search for such matrices of higher order then what we tackle with on hand. We take the first matrix shown above.

Let A = $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$; it is a R.C matrix. In two dimensional rectangular frame, the vectors $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ *and* $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are orthogonal to each other. In the same way the vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ are orthogonal to each other.

We have developed an elegant method of extending an R.C matrix of order nxn to the next higher order matrix of order (n+1)x(n+1). We shall discuss the same in the section to follow. We, just for citation purpose, write R.C matrices of order 3x3 and 4x4.

$$A = \begin{pmatrix} 1/2 & -1/2 & 1 \\ 1 & 1 & 1 \\ 1/4 & -1/2 & 1/2 \end{pmatrix} \text{ and } B = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 & -1 \\ 3 & 3 & 3 & 3 \\ 3/2 & -3 & 3 & -3 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

[**Extension of a R.C matrix from 2x2 to any R.C matrix of subsequent higher order has been shown in the annexure. Readers are requested to please go through the technical proceedings.]**

General Format of 3x3 R.C Matrix

We will be treating with a general format of 3x3 R.C matrix and derive many theorems and properties. We will consider

$$A = \begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix} \quad \textbf{(2)}$$

matrix as a standard matrix and treat this as a R.C matrix with all **real entries.**

[If all real entries are zero, then it is a null matrix and hence defining property of an R.C matrix permits a Null Matrix

in the category of a R.C matrix.

i.e $\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ -- A **null matrix** is, by definition, a R.C matrix (3)

We have, by definition of R.C matrix, the following conditions.

$a^2 + bx + cp = 0$ (4)

$bx + y^2 + qz = 0$ (5)

$cp + qz + r^2 = 0$ (6)

We shall write A = bx, B = cp, and C = qz (7)

i.e. $a^2 + A + B = 0$, $y^2 + A + C = 0$, and $r^2 + B + C = 0$

From the three equations written above, we derive

$a^2 + y^2 – r^2 = -2A \Longrightarrow A = \frac{-1}{2}(a^2 + y^2 - r^2)$. In the same way we can write two more equations. All three are listed below.

$$\left.\begin{aligned} A = bx &= \tfrac{1}{2}(-a^2 - y^2 + r^2) \\ B = cp &= \tfrac{1}{2}(-a^2 + y^2 - r^2) \\ C = qz &= \tfrac{1}{2}(a^2 - y^2 - r^2) \end{aligned}\right\} \quad (8)$$

Using the above relations those we have derived, we will prove some important notions in terms of theorems.

Theorem -1 A R.C Matrix with Real Entries, except a Null Matrix, cannot be a Symmetric Matrix

Let the R.C matrix be A = $\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}$ as defined by (2)

If it is a symmetric matrix then each one **bx**, **cp**, and **qz** must be positive but can never be negative.

Using property given in (8), as bx > 0, we have $a^2 + y^2 - r^2 < 0$

In the same way as pc > 0, we have $a^2 - y^2 + r^2 < 0$ and

qz > 0 gives $-a^2 + y^2 + r^2 < 0$

Adding all the results obtained above we have $a^2 + y^2 + r^2 < 0$ which is possible for any real values of **'a, y, and r '**

This implies that each one of bx, cp, and qz = 0. [Which can make each one of ' **a, y, and r ' = 0** and hence in turn $a^2 + y^2 + r^2 = 0$]

This proves that except a null matrix an R.C matrix cannot be symmetric one. This proves the theorem.

Lemma

On the same lines except for a non-null R.C matrix cannot be a skew-symmetric matrix.

Proof

we must have at least one of bx, pc, and qz < 0 ; while two of them are ≤ 0. This in turn, implies that, on the addition of 2bx, 2pc, and 2qz, $a^2 + y^2 + r^2 \leq 0$; which is not true. If it is a non-symmetric one then by its basic property each one of diagonal elements, **a, y, and r = 0.** This, in turn, implies that $a^2 + y^2 + r^2 < 0$; which is not possible for real entries matrix. This proves the lemma. In the next section, we will prove some defining features of 3x3 R.C matrices and then will establish that those can be extended for the R.C matrices of higher order also.

Theorem -2: A R.C Matrix, except a Null Matrix, is always a Non-Singular Matrix

Proof

- By definition, we accept that a null matrix is an R.C matrix and hence in that case, R.C matrix is a singular one.
- The case when an R.C matrix is a non-null one, by the defining property of R.C matrix we claim its non-singularity.
- [By definition of R.C matrix its i^{th} row vector is orthogonal to i^{th} column vector only making the result of their dot product/ inner product equal to zero. Had it been orthogonal to any other column except the i^{th} one then the column vectors become linearly dependent which results in the singularity of the matrix.]
- We conclude that an R.C matrix, except a null matrix, is always a non-singular matrix.

Theorem 3

- In the **3x3** R.C matrix with real entries product of at least two-off diagonal (principal) entries are negative.
- [This is an important clue to constructing a 3x3 R.C matrix. The same concept can be extended to the R.C matrices of the higher order.]

Proof

Let us consider the R.C matrix A = $\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}$ as it is defined by (2)

As mentioned in the statement we want to prove that at least **two** of the terms **bx, cp,** and **qz** must be negative; this is rather one of the most important salient features of R.C matrix. The theorem targets on establishing that at least two of **bx, cp,** and **qz are < 0**

Being an R.C matrix the entries follow R.C property. We write relations (4), (5), and (6) as below.

$a^2 + bx + cp = 0$

$bx + y^2 + qz = 0$

$cp + qz + r^2 = 0$

We introduce some notations as A = bx, B = cp, and C = qz

With this we enjoin the notations with the above relation to get

$a^2 + A + B = 0$, $y^2 + A + C = 0$, and $r^2 + B + C = 0$

From the three equations written above, we derive

$a^2 + y^2 - r^2 = -2A \Longrightarrow A = bx = \frac{1}{2}(-a^2 - y^2 + r^2)$. In the same way, we can write two more equations. All three are listed below.

$$\left.\begin{aligned} A = bx &= \tfrac{1}{2}(-a^2 - y^2 + r^2) \\ B = cp &= \tfrac{1}{2}(-a^2 + y^2 - r^2) \\ C = qz &= \tfrac{1}{2}(a^2 - y^2 - r^2) \end{aligned}\right\} \quad (9)$$

From this junction we discuss different cases for A = bx, B = cp, and C = qz.

Case 1: All of **A, B**, and **C Cannot be positive**

Say A = bx > 0. This implies that $-a^2 - y^2 + r^2 > 0$; in the same way B = cp > 0 implies that

$-a^2 + y^2 - r^2 > 0$. Adding the two results, we have $\mathbf{-2a^2 > 0}$ which is not possible.

[In the same way we can derive that $\mathbf{-2y^2 > 0}$ **and** $\mathbf{-2r^2 > 0}$**.]**

This helps us conclude that A = bx, B = cp, and C = qz all > 0 is not possible for a R.C matrix.

Case 2: Any Two of A, B, and C > 0 and the Remaining < 0 is not Possible

The proof is an immediate consequence of the above-written case 1.

Case 3: Any one of A, B, and C = 0 is not Possible

The proof is supported by case 1.mentioned above.

Case 4: Any one of A, B, and C < 0 is not Possible.

The case 2 mentioned above supports the statement and hence the proof.

Case 5: Any Two of A, B, and C are < 0 and the Remaining One is > 0.

say A = bx < 0. This implies that $-a^2 - y^2 + r^2 < 0$ and B = cp < 0. This implies that $-a^2 + y^2 - r^2 < 0$

Adding them we get $\mathbf{-2\,a^2 < 0}$ which is true. [∵ A is a real entry matrix.]

In addition to this, for C = qz > 0 implies $a^2 - y^2 - r^2 > 0$. We have to show validity of the result. From the first result we have $a^2 - r^2 > -y^2 \therefore a^2 - r^2 - y^2 > -2y^2$

$-2y^2 < 0 \Longrightarrow a^2 - r^2 - y^2 > 0$. This proves that along with bx and cp <0, C = qz > 0 is necessary.

Case 6: All the three A, B, and C are Negative.

say A = bx < 0. This implies that $-a^2 - y^2 + r^2 < 0$

B = cp < 0. This implies that $-a^2 + y^2 - r^2 < 0$

and C = qz < 0 implies that $a^2 - y^2 - r^2 < 0$

As found in the above case, adding first two relation we derived **$-2a^2 < 0$ i.e $a^2 > 0$,** we can derive

$-2r^2 < 0$ and $-2y^2 < 0$. We get each one of a^2, y^2, and $r^2 > 0$; which is true. It proves the statement.

Eigen Values

It is the most important and useful notion in matrix algebra. For the given square matrix **A** there exist a non-zero vector **X** such that for some value λ we have the matrix equation

AX = λ X satisfied. **λ** is called eigenvalue and **X i**s called the eigenvector. In this section, we discuss salient features of eigenvalues and eigenvectors for the R.C matrix.

Introduction to Important Preliminaries

Before we proceed to enunciate our findings would like to mention certain peripherals regarding the

R.C matrix and Eigenvalue. This will simplify our proceedings. All these will prove useful in the arguments proving the next theorems.

Let us initially focus our attention on A = $\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}$ a R.C matrix with all real entries defined by (2).

Let **λ_1, λ_2. and λ_3** be the eigen values of **A** with corresponding non zero eigen vectors **X_1, X_2, and X_3.**

(1) As the matrix A is an R.C matrix, by defining properties, we have the following results. These results are already mentioned in earlier work but just to abridge we cite those at this point.

$a^2 + bx + cp = 0$, $bx + y^2 + qz = 0$, and $cp + qz + r^2 = 0$

We introduce some notations as A = bx, B = cp, and C = qz and derive

$$\left.\begin{aligned} A &= bx = \tfrac{1}{2}(-a^2 - y^2 + r^2) \\ B &= cp = \tfrac{1}{2}(-a^2 + y^2 - r^2) \\ C &= qz = \tfrac{1}{2}(a^2 - y^2 - r^2) \end{aligned}\right\} \therefore \quad a^2 + y^2 + r^2 = -2\,(bx + cp + qz)$$

(2) Also recalling the facts pertaining to eigen values **λ_1, λ_2, and λ_3; we write**

(a) Sum of Eigen values = **$\lambda_1 + \lambda_2 + \lambda_3$ = Trace of the matrix = a + y + r** (10)

(b) Sum of product of eigen values taken two at a time= **$\lambda_1 \lambda_2 + \lambda_3 \lambda_3 + \lambda_2 \lambda_3$= (ay-bx)+(ar-pc)+(yr-qz)** (11)

(c) Product of Eigen values = **$\lambda_1 \lambda_2 \lambda_3$ = det. A = |A|** (12)

With this on hand we state some theorems.

Theorem 4: Eigen values of an R.C Matrix with Real Entries are either all Zero or exactly One Real.

Proof: As a null matrix is also an R.C matrix, we have all eigenvalues are zero and hence the proof

If the matrix **A** is not a null matrix then we proceed as follows.

Using the facts mentioned we have $(\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1 \lambda_2 + \lambda_3 \lambda_3 + \lambda_2 \lambda_3)$

$\therefore (a + y + r)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(ay-bx)+(ar-pc)+(yr-qz)$

but $a^2+y^2+r^2 = -2(bx+ cp +qz)$ [as mentioned earlier]

So we have $(a + y + r)^2 = -2(bx+ cp +qz) + 2(ay +yr + ar)$

On comparing two results for $(a + y +r)^2$ we have

$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(ay-bx)+(ar-pc)+(yr-qz) = -2(bx+ cp +qz) + 2(ay +yr + ar)$

$\Rightarrow \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$ and hence the proof. (13)

Theorem 5: All the Eigen Values of A R.C Matrix are Non-Zero. (14)

Proof: If any one of the eigen value is zero then it implies that $|A| = 0$.

This means that the R.C matrix is a singular matrix. This violates the defining property of the R.C matrix. It, except the null matrix, is always non-singular.

[Note: We have the derived fact that the eigen values λ_1**,** λ_2**, and** λ_3 are such that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$ **then one is *real and different from zero* while other two are complex conjugate of each other.]**

Important Derivation

In tune with the above results, we have up till now two important results--- **(13)** and **(14)**

From (13) we write that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$ with no λ being zero.

$\Rightarrow \lambda_2^2 + \lambda_3^2 = -\lambda_1^2$. We conclude that λ_2**,** and λ_3 are complex conjugates of each other.

Let $\lambda_2 = \propto + i\beta$ and $\lambda_3 = \propto - i\beta$. So we get $\lambda_2^2 + \lambda_3^2 = 2(\alpha^2 - \beta^2) = -\lambda_1^2$

$\therefore -\lambda_1^2 = 2(\alpha^2 - \beta^2)$ **(11)**

This logically implies that $|\beta| > |\alpha|$

These eigenvalues will satisfy $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$

Deductions: The results established above will help deduce the following relations.

For the R.C matrix A = $\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}$

$$\left.\begin{array}{ll} \textbf{1 Trace} = \lambda_1 + \lambda_2 + \lambda_3 = a + y + r = \mathbf{T} \text{ say} & \textbf{(a)} \\ \mathbf{2}\ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = \mathbf{(ay-bx) + (ar-cp) + (yr-qz) = M} \text{ say} & \textbf{(b)} \\ \mathbf{3}\ \lambda_1 \lambda_2 \lambda_3 = \mathbf{|A| = det.\ A = D} & \textbf{(c)} \\ \mathbf{4}\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \mathbf{0} & \textbf{(d)} \end{array}\right\} \quad \textbf{(12)}$$

Using all these we derive

5 $\lambda^3 - T\lambda^2 + M\lambda - D = 0$ Characteristic equation

6 $\lambda_2 + \lambda_3 = 2\alpha = T - \lambda_1$ and $\lambda_2 - \lambda_3 = 2i\beta$ and in connection with (a) above

7 $\lambda_1 = T - 2\alpha$ i.e $\alpha = (\lambda_1 - T)/2$ and $\lambda_1(2\alpha) + \lambda_2\lambda_3 = M$ (13)

8 Using $(\lambda_2 - \lambda_3)^2 = 2\sqrt{(\alpha^2 - \frac{D}{\lambda 1})}$ we have $\lambda_2 = \alpha + \sqrt{(\alpha^2 - \frac{D}{\lambda_1})}$ and $\lambda_3 = \alpha - \sqrt{(\alpha^2 - \frac{D}{\lambda_1})}$

This conveys that knowing only the real eigenvalue it is sufficient enough to write the remaining two complex conjugate eigenvalues.

9 Fact: in a given R.C matrix there exists at least one column C_k or a row say R_k such that for a non –zero real value ‘c’ such that either $C_k = c.\ C_{k1}$ or $R_k = c.\ R_{k1}$; i.e. the column or the row is a multiple of some real constant.

For $A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 2 & -6 \\ 1 & 1 & 3 \end{pmatrix}$ the third column $C_3 = 3\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

(3) Graphical Method of Approximating Real Eigen Value

By now, it is well known that a non-null R.C matrix has only one non-zero real root while the remainder two are complex conjugate. [At this stage we reiterate that A real entry R.C matrix cannot be either symmetric or skew symmetric.]

The vision to shape this section is to locate graphically and approximate algebraically the real root of the characteristic equation of the given R.C matrix. As we have discussed many possible properties

Inter-linking the different eigenvalues of a given R.C matrix, we state here what we shall require at times. We need the first one (5) above in set (12); It is our characteristic equation.

$\lambda^3 - T\lambda^2 + M\lambda - D = 0$

For real eigen root **λ,** the graph of f(**λ**) on a set of perpendicular real axis, will intersect the x-axis in a point, say $x_1 =$ **λ_1;** its location is our objective.

For $\lambda = 0$, $f(\lambda) = -D$ where **$D = \det.|A| = \lambda_1\lambda_2\lambda_3$** where, as said earlier, **λ_2, and λ_3** are complex eigen values.

Plotting this, we get the graph of a cubic curve.

We parallel our work citing a real R.C matrix.

Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 2 & -6 \\ 1 & 1 & 3 \end{pmatrix}$ With T = Trace = 6, M= 18, and D = -3

$f(\lambda) = \lambda^3 - 6\lambda^2 + 18\lambda + 3$, for $\lambda = 0$, $f(\lambda) = 3$. This situation is graphed as below in figure-1.The figure -2 shows its magnification on an interval about its intersection on x axis.

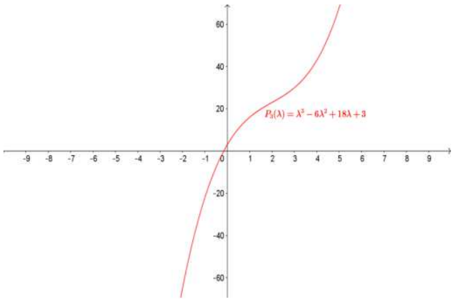

Figure 1

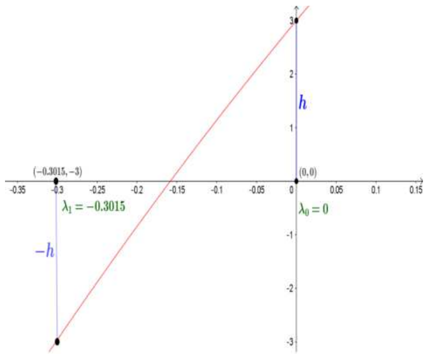

Figure 2: (Magnified Interval)

Let f(0) = h [In our case f(0)= 3]. As it has root, it crosses x-axis. This implies that there exists x_2 [$x_2 < 0$] such that $f(x_2)$ = -h. [In our case $f(x_2)$ = –h = -3] We can always find such x_2 algebraically.

As $f(x_2 = - 0.3015) = -3$ and f(0)= 3, root = x_3 lies in (- 0.3015,0),

$f(x_2) = -h < f(0)= h$; $x_3 \in (x_2, 0)$. Let $x_3 = (x_2 + 0)/2.= x_2 /2$, Now we find $f(x_3)$

The next approximation is (-0.3015 / 2 = - 0.15075), f(-0.15075) = h_1 say.

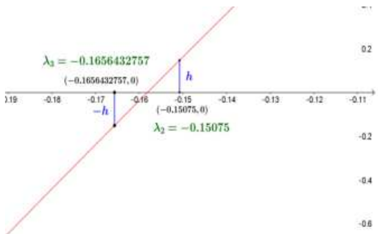

Figure 3: (Iterative Version)

In this way after a finite number of iteration, for a given $\epsilon > 0$, we can find a real x_n so that $|f(x_n/2| < \epsilon$.This is the most effective method for approximating graphically the finer approximation to the real eigenvalue.

Vision

During the time that we derived and critically reviewed the characteristics of R.C matrices of dimensions 2x2 and onwards, we could find many interesting features. We commit, we have searched a small area and still, we enjoin our efforts inspired by a new result we work upon. Excavating such unknown area may elaborate mathematically ignited minds.

All constructive suggestions are welcome.

Annexure

As discussed, we, in this section, will elaborate the technique of finding an extension of a 2x2 R.C matrix to the R.C matrices of the higher order. We begin with a simple R.C matrix of order 2x2.

Let $A_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Let us consider the column system as $y_1 = 1x + 1$ and $y_2 = -1x + 1$ [which shows perpendicular lines in R^2 space.]

Integrating each one with respect to x, we get

$y_{11} = x^2/2 + x + c_1$ and $y_{21} = - x^2/2 + x + c_2$.

The matrix which corresponds to this system is

$A_2 = \begin{pmatrix} 1/2 & -1/2 \\ 1 & 1 \\ c_1 & c_2 \end{pmatrix}$ We extend this matrix A_2 as $A_2 = \begin{pmatrix} 1/2 & -1/2 & p \\ 1 & 1 & q \\ c_1 & c_2 & c_3 \end{pmatrix}$ where all the letters in different positions are the real values. It is so planned that they satisfy R.C. property.

We have, $(1/2)^2 + (-1/2)(1) + pc_1 = 0$, $(-1/2)(1) + 1 + qc_2 = 0$, and $pc_1 + qc_2 + (c_3)^2 = 0$

This gives us a free choice for selection of variables remaining within the given equation.

We select $p = 1$, $c_1 = 1/4$, $q = 1$, $c_2 = -1/2$ and hence $c_3 = ½$.

The extended version of R.C matrix is now, $A_2 = \begin{pmatrix} 1/2 & -1/2 & 1 \\ 1 & 1 & 1 \\ 1/4 & -1/2 & 1/2 \end{pmatrix}$

Again on the same lines, this can be extended to an R.C matrix of the size 4x4.

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